A SIMPLE APPROACH TO EVALUATE THE LINEWIDTH OF A LASER FROM ITS FREQUENCY NOISE SPECTRAL DENSITY

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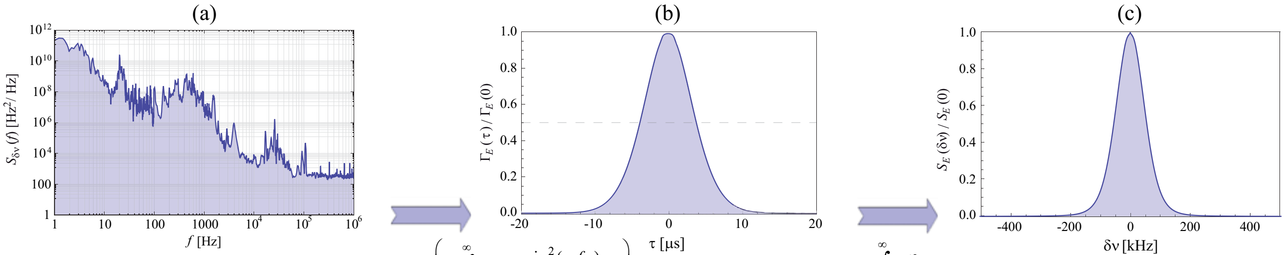
1) INTRODUCTION

The theoretical relation between the frequency noise power spectral density (PSD) of a laser and its optical lineshape is known for a long time [1] and requires a 2-step integration as illustrated in Fig. 1. The first step calculates the auto-correlation function of the electrical field $\Gamma_E(\tau)$, from which the optical spectrum $S_E(\nu)$ is obtained in the second step by Fourier transform. A more detailed theoretical description of this procedure will be given in a separated paper [2]. However, this procedure is far to be straightforward and must be performed by numerical integration for real laser frequency noise spectra encountered in practice. As the linewidth is a simple commonly-used parameter to characterize and compare laser spectral properties, it would be useful to experimentalists to have a simpler way to determine the linewidth of a laser from its arbitrary frequency noise PSD.



$$\Gamma_E(\tau) = E_0^2\, e^{2\pi i\nu_0\tau} \exp\left(-2\int_0^\infty S_{\delta\nu}(f)\frac{\sin^2(\pi f\tau)}{f^2}df\right) \qquad S_E(\nu) = 2\int_{-\infty}^{\infty} e^{-i2\pi\nu_0\tau}\,\Gamma_E(\tau)d\tau$$

Fig. 1: Schematic description of the theroretical route from the frequency noise power spectral density $S_{\delta\nu}(f)$ (a) to the lineshape function $S_E(\nu)$ (c) via the auto-correlation function $\Gamma_E(\tau)$ of the electric field (b).

In this work, we present a straightforward approach to make this link using a simple geometric formula to determine the linewidth corresponding to an arbitrary noise spectrum. We start with the ideal case of a low-pass filtered white noise of varying cutoff frequency to show the different impact of the low- and high-frequency noise components to the lineshape. This leads to the introduction of our simple approximation of the linewidth based on a geometrical separation of the noise spectrum into two areas with a fully different influence on the lineshape. By showing how some spectral components of the noise determine the linewidth while others only affect the wings of the lineshape, we provide a simple geometric criterion to determine the spectral components contributing to the linewidth. Then we apply this formula to a situation of practical interest to experimentalists, which is the reduction of the linewidth of a laser with a servo-loop. Finally, we illustrate this particular case with some experimental results.

2) LOW-PASS FILTERED WHITE NOISE

The simple case of a constant frequency noise spectral density of level h_0 [Hz^2/Hz] below a cutoff frequency f_c as shown in Fig. 2a is an instructive example for which an analytical expression of the lineshape can be analytically obtained in the two extreme conditions where $f_c/h_0 \rightarrow \infty$ and $f_c/h_0 \rightarrow 0$. A Lorentzian lineshape is obtained with a full width at half maximum of FWHM = πh_0 when $f_c/h_0 \rightarrow \infty$, while a Gaussian lineshape with a width of FWHM = $h_0\sqrt{8\ln(2)\, f_c/h_0}$ that depends on the cutoff frequency f_c is obtained when $f_c/h_0 \rightarrow 0$. Fig. 2c shows the evolution of the laser lineshape numerically calculated for different cutoff frequencies f_c for a fixed frequency noise level h_0. As previously stated, one observes that when $f_c << h_0$ the lineshape is Gaussian and the linewidth increases with f_c. However, when $f_c >> h_0$, the lineshape becomes Lorentzian and the linewidth stops to increase. The transition between these two regimes was numerically explored by calculating the linewidth as a function of the cutoff frequency f_c (see Fig. 3). A good approximation of the linewidth valid for any f_c is given by:

$$\mathrm{FWHM} = h_0 \sqrt{8\ln(2)\frac{f_c}{h_0}} \Bigg/ \sqrt[4]{1+\left(\frac{8\ln(2)}{\pi^2}\frac{f_c}{h_0}\right)^2} \,,$$

with a relative error smaller than 4% over the entire range of the cutoff frequency f_c as displayed in Fig. 3b. The corner frequency corresponding to the transition between the two regimes is situated at the intersection of the two asymptotes shown in Fig. 3a and is given by:

$$f_c^* = \frac{\pi^2}{8\ln(2)} h_0 \,.$$

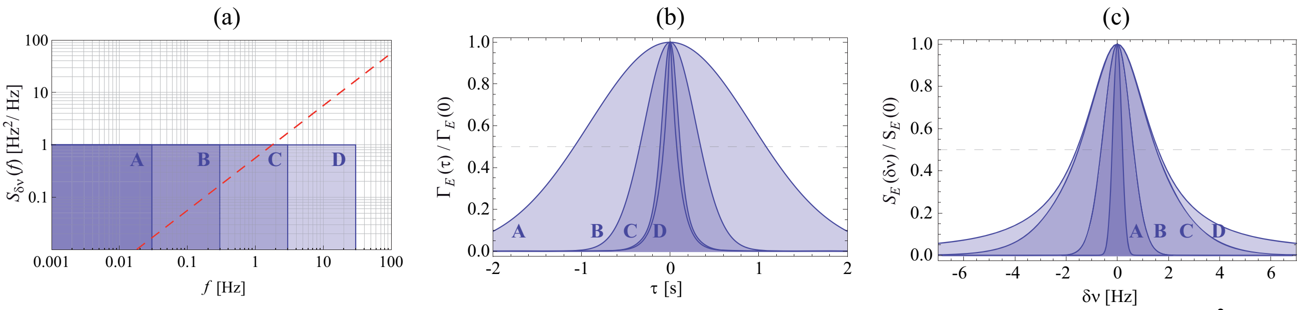


Fig. 2: Simple case of a low-pass filtered white frequency noise. (a) Frequency noise PSD with a level $h_0 = 1$ Hz^2/Hz and four different values of the cutoff frequency f_c: (A) f_c = 0.03 Hz; (B) f_c = 0.3 Hz; (C) f_c = 3 Hz; (D) f_c = 30 Hz. (b) Corresponding auto-correlation function and (c) lineshape function displayed for the different values of f_c.

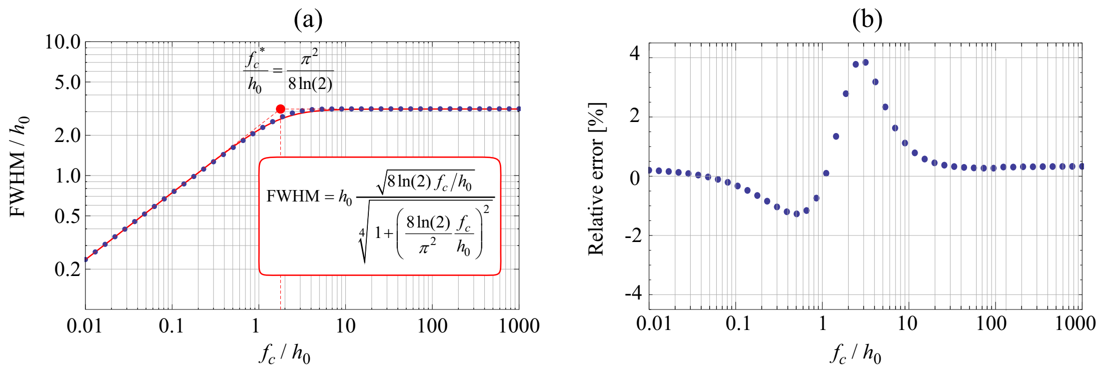


Fig. 3: (a) Evolution of the linewidth as a function of the normalized cutoff frequency f_c/h_0 given by our approximate analytical expression (red line corresponding to the equation in the inset) and numerically calculated (blue points). (b) Relative error between the actual numerically calculated linewidth and the approximated value.

3) A SIMPLE FORMULA TO ESTIMATE THE LINEWIDTH

The example of the low-pass filtered white noise shows that the frequency noise spectrum is separated into two parts which affect the lineshape in a very different way. When $S_{\delta\nu}(f) > 8\ln(2) f / \pi^2$ (corresponding to a high frequency modulation (FM) index), the frequency noise contributes to the laser linewidth, while the regions for which $S_{\delta\nu}(f) < 8\ln(2) f / \pi^2$ (low FM index) contribute only to the wings of the lineshape and do not impact the linewidth. This observation enables us to separate the spectrum into different areas delimited by the line $S_{\delta\nu}(f) = 8\ln(2) f / \pi^2$ that we call the "magic" line. A good approximation of the linewidth is inferred from an integration of the frequency noise spectrum restricted over those values of the Fourier frequency for which the frequency noise is higher than the "magic" line, corresponding to the surface A of the high FM index area:

$$\mathrm{FWHM} = \sqrt{8\ln(2) A} \,.$$

Fig. 4 illustrates the situation of a typical laser frequency noise spectral density composed of low-frequency flicker noise and high-frequency white noise. In this specific case, as well as in any case showing a monotonously decreasing frequency noise, the upper integration limit for the determination of the linewidth is given by the intersection of the frequency noise spectrum with the "magic" line.

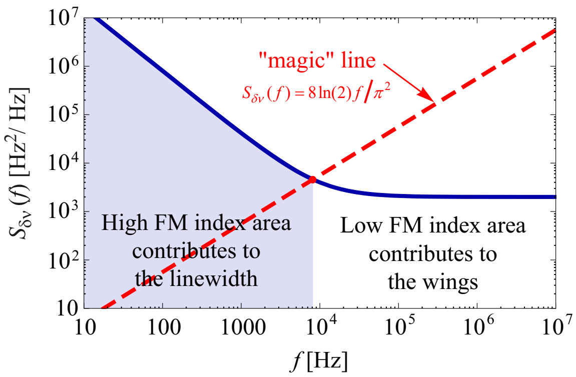


Fig. 4: The "magic" line given by $S_{\delta\nu}(f) = 8\ln(2) f / \pi^2$ separates the spectrum into two areas with a different contribution to the laser lineshape: the high FM index area of surface A contributes to the linewidth whereas the low FM index area contributes only to the wings of the lineshape.

4) APPLICATION TO THE REDUCTION OF THE LINEWIDTH OF A LASER

As an illustration, we apply our approach to a simplified process of laser linewidth reduction using a servo-loop. We consider a free-running laser with a constant frequency noise level h_b [Hz^2/Hz] that is reduced to a lower level h_a with a feedback loop of bandwidth f_b as illustrated in Fig. 5a. In that case, our geometrical approximation of the laser linewidth is compared in Fig. 5b with the value deduced from the numerically calculated lineshape. We notice that the linewidth tends toward πh_b when the bandwidth tends toward zero and drops down to πh_a when the bandwidth tends toward infinity. The agreement between the geometrical approximation and the numerical integration is good except when the value of the servo bandwidth is between h_a and h_b. The reason is that when the frequency noise approaches the "magic" line, the servo loop repels the frequency noise from the center of the lineshape to create sidebands out of the servo bandwidth, which radically changes the lineshape, whose central part strongly narrows. This behavior is poorly accounted for by the geometrical approximation which loses its significance in this range. However, our simplified approach provides a good approximation of the servo-loop bandwidth required to efficiently reduce the laser linewidth, given by $f_b^{min} = \pi^2 h_b / 8\ln(2)$. Furthermore, it shows that for higher bandwidths, the achievable laser linewidth is determined only by the low frequency noise level, thus by the low frequency gain of the servo-loop.

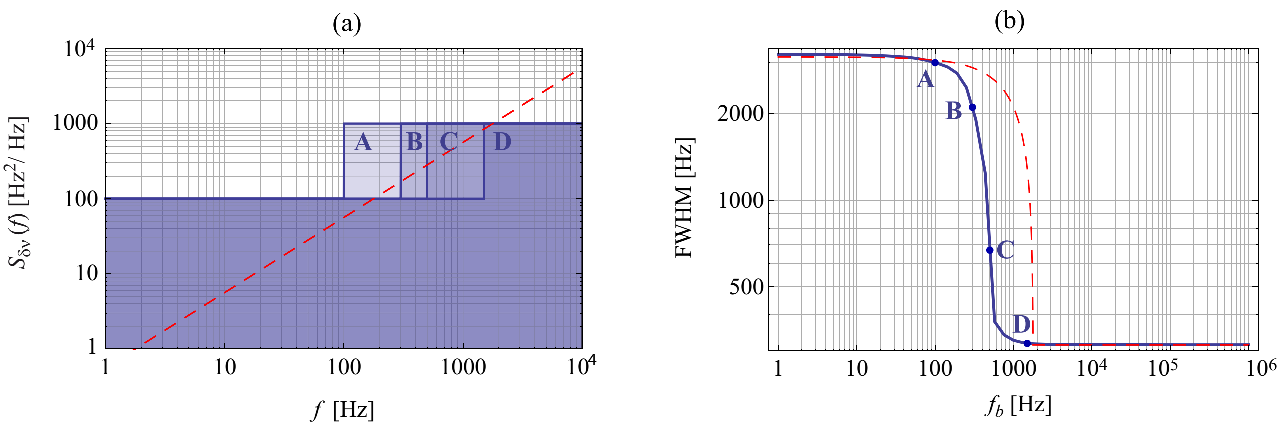


Fig. 5: Simplified model of laser linewidth reduction using a servo-loop. (a) Frequency noise model assuming a constant noise level h_b = 1000 Hz^2/Hz of the free-running laser that is reduced to a level h_a = 100 Hz^2/Hz with a servo-loop with a varying bandwidth f_b = 100 Hz (A), f_b = 300 Hz (B), f_b = 500 Hz (C), f_b = 1500 Hz (D). (b) Evolution of the laser linewidth as a function of the servo loop bandwidth f_b. The continuous blue line has been obtained by numerical integration and the dashed line corresponds to our approximate formula.

5) EXPERIMENTAL RESULTS

We show here some experimental results to illustrate our simplified theoretical approach of the relation between the frequency noise PSD and the linewidth. We do not consider here a laser with its optical spectrum, but the carrier envelope offset (CEO) beat signal from an optical frequency comb. However, the same formalism applies. A detailed description of the used frequency comb with its stabilization is reported in another paper of these conference proceedings [3]. We just recall here that the 20-MHz CEO-beat signal may be stabilized to an external reference using a phase-lock loop (PLL). The frequency noise spectral density of the CEO-beat was determined using a frequency discriminator independent of the CEO stabilization loop. This frequency discriminator, based on a PLL, has a sensitivity of 0.7 V/MHz and a bandwidth of several hundreds of kilohertz. Additionally, the RF spectrum of the CEO-beat was measured using an electrical spectrum analyzer to extract the corresponding linewidth. Results are presented in Fig. 6 for the three following situations: (a) free-running CEO; (b) CEO weakly locked and (c) CEO tightly locked to the external reference.

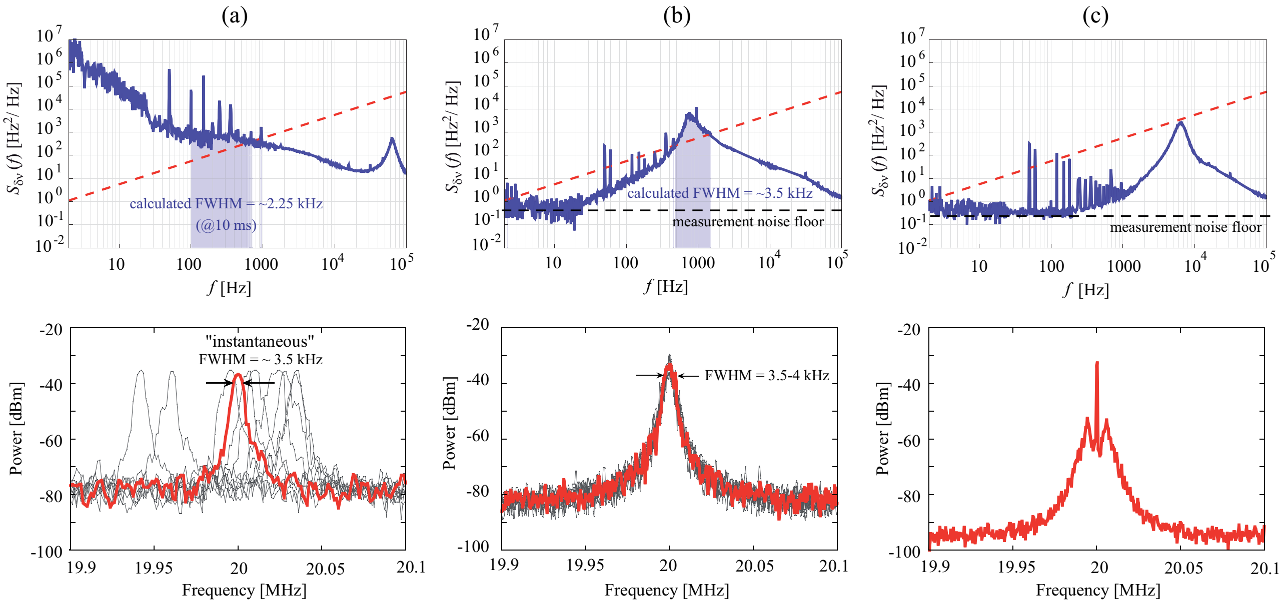


Fig. 6: Comparison of the frequency noise PSD (upper graphs) and RF spectra (lower graphs) of the CEO-beat obtained in the three following situations: (a) free-running CEO; (b) CEO weakly locked and (c) CEO tightly locked to the external reference. When the frequency noise PSD completely passes below the "magic" line (red dashed line), the linewidth dramatically drops to virtually zero and a delta function appears at the carrier frequency in the RF spectrum, superimposed to a larger pedestral due to the residual phase noise (case c). For cases (a) and (b), ten successively recorded RF spectra are shown in grayscale to demonstrate the difference in the stability of these two CEO-beat signals (see text for detail).

For the free-running CEO-beat (case a), the frequency noise spectrum is dominated by flicker noise at low frequency, leading to a linewidth that strongly depends on the observation time. A linewidth of ~2.25 kHz at 10 ms is determined from our geometrical approach, which is in reasonable agreement with the ~3.5 kHz linewidth experimentally observed, taking into account that the experimental observation time is poorly known. In case (b), the stabilization loop strongly reduces the low-frequency noise below the "magic" line at Fourier frequencies smaller than ~500 Hz, thus removing their contribution to the linewidth. At Fourier frequencies smaller than ~30 Hz, the frequency noise measurement is limited by the noise floor of our frequency discriminator, which is at the level of ~0.4 Hz^2/Hz. However, the frequency noise is expected to continue to decrease at lower frequencies due to the increasing gain of the CEO servo-loop. A bump of increased noise is observed in the frequency noise spectrum in the range 500-2000 Hz. This bump surpasses the "magic" line and thus contributes to the linewidth with an area A = ~2.2·10^6 Hz^2. This gives rise to an approximated linewidth of ~3.5 kHz, in qualitative agreement with the 3.5-4 kHz linewidth experimentally observed. This linewidth is paradoxically slightly larger than the short-term linewidth of the free-running CEO, even if the CEO-beat is locked in this second case. But the major difference between cases (a) and (b) is that the beat signal obtained for the locked CEO (b) is fully stable, reproducible and independent of the observation time as shown by ten successively recorded traces on the RF spectrum analyzer and displayed in Fig. 6b. On the opposite, the free-running CEO-beat strongly fluctuates so that successive measurements severely differ from each other as shown by the ten strongly different traces displayed in Fig 6a. As a consequence, the linewidth strongly depends on the observation time. In case (c), the increased gain of the servo-loop still reduces the noise level which becomes inferior to the "magic" line at all Fourier frequencies (the measurement below a few hundred hertz is limited again by the noise floor of our discriminator). A bump in the spectrum is also observed around 6.5 kHz, which is believed to result from the limited bandwidth of the fs-laser due to the lifetime of the excited energy level of the Er ions in the gain medium. Anyway, this bump remains below the "magic" line, as all other spectral components, and thus does not contribute to the linewidth. The linewidth becomes infinitely narrow and a delta function appears at the carrier frequency in the RF spectrum, superimposed to a larger pedestral due to the residual phase noise. This is experimentally confirmed by the presence of the central narrow peak with >20 dB signal-to-noise ratio in a 30-Hz resolution bandwidth in the centre of the RF spectrum displayed in Fig. 6(c). An additional high resolution measurement of this central peak showed that its width was instrument-limited to < 4 mHz.

6) CONCLUSION

We have shown how the frequency noise spectrum is separated into different areas (corresponding to high and low FM index regimes), delimited by a simple "magic" line. Only those spectral components for which the frequency noise is higher than the "magic" line (high FM index area) contribute to the linewidth, whereas other frequency components only affect the wings of the lineshape. From these observations, an approximate value of the linewidth which is useful to experimentalists has been deduced and is simply obtained from the geometrical surface of the high FM index area.

We have illustrated this approach by a situation of practical interest to experimentalists, which is the reduction of the linewidth of a laser using a servo-loop. Our simple approach emphasizes some important aspects of this problem, such as the minimal required servo-loop bandwidth and the achievable laser linewidth. Finally, we have shown experimental results that demonstrate the importance of the "magic" line in the frequency noise spectrum and the dramatic drop of the linewidth to virtually zero that occurs when the frequency noise PSD completely falls down below the "magic" line. This behaviour is revealed by the appareance of a delta function at the carrier frequency in the RF spectrum.

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